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The Cauchy problem for heat equations with exponential nonlinearity

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ABSTRACT

The Cauchy problem for the semilinear heat equations is studied in the Orlicz space $\exp L^2(\mathbb{R}^n)$, where any power behavior of interaction works as a subcritical nonlinearity. We prove the existence of global solutions for the semilinear heat equations with the exponential nonlinearity under the smallness condition on the initial data in $\exp L^2(\mathbb{R}^n)$.

1. Introduction and main results

In this paper we consider the Cauchy problem for a semilinear heat equation with exponential nonlinearity:

$$\begin{cases} \partial_t u - \Delta u = f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u(t,x) : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, u_0 is a given initial data and $f(u) \sim e^{u^2}$ for $|u| \geqslant 1$.

It is known that if the initial data is bounded then there exist $T > 0$ and a unique local solution in the class of $C((0,T); L^\infty(\mathbb{R}^n))$ of (1.1) (cf. Ladyzhenskaya, Solonnikov and Ural'tseva [12]). The first

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existence result for the unbounded data is due to Weissler [26,27] (see also Brézis and Cazenave [2] and Giga [5]). He considered the Cauchy problem with the power nonlinearity:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

with initial data $u_0 \in L^r(\mathbb{R}^n)$, $p > 1$ and $1 < r < \infty$. We see that if the function $u(t,x)$ satisfies Eq. (1.2), then for any $\lambda > 0$, the scaled function $u_\lambda(t,x) = \lambda^{2/(p-1)}u(\lambda^2 t, \lambda x)$ also satisfies (1.2) and L^r norm of the solution is invariant under this scaling if and only if $p = p_h^* := 1 + \frac{2r}{n}$. This exponent p_h^* plays a crucial role for the existence result of the Cauchy problem (1.2). Indeed, there exist $T = T(u_0)$ and a unique local solution $u \in C([0,T); L^r(\mathbb{R}^n))$ of the Cauchy problem (1.2) if $1 < r < \infty$ and $1 < p \leqslant p_h^* = 1 + \frac{2r}{n}$. On the other hand, if $p > p_h^*$, then there exists no local solution in any reasonable sense for general initial data $u_0 \in L^r(\mathbb{R}^n)$. In the critical case $p = p_h^*$, the solution in $L^r(\mathbb{R}^n)$ exists globally in time for small initial data in particular.

Cazenave and Weissler [3] mentioned that there exists a certain relationship between the L^r theory for the nonlinear heat equation and the H^s theory for the nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

Cazanave and Weissler showed that for $0 \leqslant s < \frac{n}{2}$ there exists a critical exponent $p_s^* = 1 + \frac{4}{n-2s}$ that has a similar scaling invariance in $\dot{H}^s(\mathbb{R}^n)$ for (1.3) such that for every $1 < p \leqslant p_s^*$ there exist $T = T(u_0) > 0$ and a unique solution $u \in C([0,T); H^s(\mathbb{R}^n))$ of the Cauchy problem (1.3). Moreover, the global existence for small data can be also proved for the critical case $p = p_s^*$. We should mention that if we choose $s = \frac{n}{2}$, then the critical nonlinearity should be stronger than any power nonlinearity since $p_s^* = \infty$. Indeed, Nakamura and Ozawa [15] showed the small data global existence in $H^{\frac{n}{2}}(\mathbb{R}^n)$ of the nonlinear Schrödinger equation with exponential nonlinearity $f(u) \simeq e^{u^2}$ ($|u| \geqslant 1$). For related results, see [8,10,14,15,18,23] and references therein.

Cazanave and Weissler suggested that the critical exponent p_h^* of the heat equation (1.2) and the critical exponent p_s^* of the Schrödinger equation (1.3) are connected by the Sobolev embedding, $\dot{H}^s(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$, where s, r satisfy $0 \leqslant s < \frac{n}{2}$ and $\frac{1}{r} = \frac{1}{2} - \frac{s}{n}$. It is easy to see that the embedding implies the relation $p_h^* = 1 + \frac{2r}{n} = 1 + \frac{4}{n-2s} = p_s^*$, hence the functional space for the critical exponents of both (1.2) and (1.3) coincides in view of the Sobolev embedding. Ruf and Terraneo [21] proposed as the analogy of the relation mentioned by Cazanave and Weissler that the small data global existence of the heat equation (1.1) with the exponential nonlinearity $f(u) \simeq e^{u^2}$ ($|u| \geqslant 1$) can be proved in a space which includes $H^{\frac{n}{2}}(\mathbb{R}^n)$ by the Sobolev embedding. It is known that there exists no suitable space in the Lebesgue spaces which includes $H^{\frac{n}{2}}(\mathbb{R}^n)$ in order to solve (1.1) with the exponential nonlinearity. The better space is given by the Orlicz space $\exp L^2(\mathbb{R}^n)$ which is a generalization of Lebesgue spaces.

Definition 1.1. The space $\exp L^2(\mathbb{R}^n)$ is the set of all functions which satisfies

$$\int_{\mathbb{R}^n} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx < \infty \quad \text{for some } \lambda > 0.$$

The space $\exp L^2(\mathbb{R}^n)$ is the Banach space by the Luxemburg norm:

$$\|u\|_{\exp L^2(\mathbb{R}^n)} := \inf\left\{ \lambda > 0;\ \int_{\mathbb{R}^n} \left(\exp\left(\frac{|u(x)|}{\lambda} \right)^2 - 1 \right) dx \leqslant 1 \right\}.$$

It is known that from Trudinger–Moser's inequality [13,16,17,24] the Orlicz space $\exp L^2(\mathbb{R}^n)$ has the embeddings $H^{\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow \exp L^2(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ for all $2 \leqslant r < \infty$. We should mention that $\exp L^2(\mathbb{R}^n)$ is the smallest space which includes $H^{\frac{n}{2}}(\mathbb{R}^n)$ in the framework of Orlicz spaces.

Ruf and Terraneo considered the heat equation (1.1) with the following nonlinear term f: For every $x, y \in \mathbb{R}$,

$$\left|f(x) - f(y)\right| \leqslant \begin{cases} C|x-y|(|x|e^{\lambda x^2} + |y|e^{\lambda y^2}), & n \geqslant 4, \\ C|x-y|(x^2e^{\lambda x^2} + y^2e^{\lambda y^2}), & n = 2, 3, \\ C|x-y|(x^4e^{\lambda x^2} + y^4e^{\lambda y^2}), & n = 1, \end{cases} \tag{1.4}$$

for some $C > 0$ and $\lambda > 0$. They showed the local existence of a solution for the heat equation (1.1) with (1.4) for small initial data in $\exp L^2(\mathbb{R}^n)$. Our aim in this paper is to prove the small data global existence for the Cauchy problem (1.1) with (1.4) in $\exp L^2(\mathbb{R}^n)$. Let $e^{t\Delta}$ be the heat evolution operator given by

$$e^{t\Delta}u_0(x) := \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy.$$

Although the evolution operator $e^{t\Delta}$ is bounded from $\exp L^2(\mathbb{R}^n)$ to $\exp L^2(\mathbb{R}^n)$, it is not a contraction semigroup. Indeed, $e^{t\Delta}$ is not continuous at $t = 0$ in $\exp L^2(\mathbb{R}^n)$ since the space of smooth functions with compact support $C_0^\infty(\mathbb{R}^n)$ is not dense in $\exp L^2(\mathbb{R}^n)$. Namely, for some initial data u_0 and positive constant $C > 0$, we have

$$\lim_{t \to 0} \left\| e^{t\Delta}u_0 - u_0 \right\|_{\exp L^2(\mathbb{R}^n)} \geqslant C. \tag{1.5}$$

Therefore we consider the initial value problem (1.1) in the following weak sense:

Definition 1.2 *(Weak mild solutions).* We say that $u \in L^\infty(0, \infty; \exp L^2(\mathbb{R}^n))$ is a weak mild solution for the Cauchy problem (1.1) with (1.4) if u satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f\big(u(s)\big)\, ds \tag{1.6}$$

in $\exp L^2(\mathbb{R}^n)$ for almost all $t > 0$ and u satisfies continuity to the initial data in the following sense:

$$\text{w}^*\text{-}\lim_{t \to 0} u(t) = u_0 \quad \text{in } \exp L^2\big(\mathbb{R}^n\big). \tag{1.7}$$

The global existence of the solution for the problem (1.1) in Lebesgue spaces has not been clear since the nonlinear term which satisfies (1.4) grows essentially faster than any power type nonlinearities. Using the framework of the Orlicz space, we obtain the global existence for the Cauchy problem (1.1) with (1.4).

Theorem 1.3. *There exists a positive constant $\varepsilon > 0$ such that for every initial data $u_0 \in \exp L^2(\mathbb{R}^n)$ which satisfies $\|u_0\|_{\exp L^2(\mathbb{R}^n)} \leqslant \varepsilon$, there exists a weak mild solution $u \in L^\infty(0, \infty; \exp L^2(\mathbb{R}^n))$ of the Cauchy problem* (1.1) *with* (1.4)*.*

Remark 1.4. Many authors studied the equation

$$\partial_t u - \Delta u = e^u$$

in different view points. See for instance [4,6,19,22,25] and references therein. They showed the global existence result under the pointwise assumption for the initial data u_0, namely $u_0(x) \leqslant U(x)$, where U is the singular solution of the stationary problem $-\Delta u = e^u$. Under this assumption, the initial data does not have singularities except at the origin. We should mention that they also considered the uniqueness and blow-up problems. Their method is mainly based on the comparison principle. We do not use the comparison principle in the proof of Theorem 1.3. This suggests a possibility that Theorem 1.3 can be extended to a problem of system of equations. Moreover, Theorem 1.3 is proved under the smallness assumption on the initial data u_0 in $\exp L^2(\mathbb{R}^n)$, hence it is possible to choose the initial data which has some singularities not only at the origin but also at any other points.

We prove Theorem 1.3 by a contraction mapping argument. Hence the uniqueness result holds in the space where a contraction mapping argument is applied. Moreover, the solution satisfies (1.1) in classical sense if $t > 0$. We mention that the assumption for the nonlinear term of Theorem 1.3 covers the nonlinearity of the form

$$f(u) = \begin{cases} e^{u^2} - 1, & n \geqslant 4, \\ (e^{u^2} - 1)u, & n = 2, 3, \\ (e^{u^2} - 1 - u^2)u, & n = 1. \end{cases}$$

We also remark that the solution satisfies the following property:

$$\lim_{t \to 0} \left\| u(t) - e^{t\Delta} u_0 \right\|_{\exp L^2(\mathbb{R}^n)} = 0. \tag{1.8}$$

This property means that the nonlinear term of the problem (1.6) converges to 0 as $t \to 0$ in $\exp L^2(\mathbb{R}^n)$. One can find this type continuity property in Quittner and Souplet [20].

This paper is organized as follows. In Section 2, we show some basic properties of the evolution operator $e^{t\Delta}$ in $\exp L^2(\mathbb{R}^n)$. We prove Theorem 1.3 for higher dimensions $n \geqslant 5$ in Section 3 and for lower dimensions $1 \leqslant n \leqslant 4$ in Section 4. We also prove the claim of the inequality (1.5), (1.7) and (1.8) which means continuity of the solution at $t = 0$ in Section 5.

2. Preliminary

For the proof, we prepare some basic linear estimates.

Lemma 2.1. *(See [21].) For every $2 \leqslant p < \infty$, the following inequality holds:*

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant \left\{ \Gamma\left(\frac{p}{2} + 1\right) \right\}^{\frac{1}{p}} \|u\|_{\exp L^2(\mathbb{R}^n)}, \tag{2.1}$$

where Γ is the gamma function

$$\Gamma(p) := \int_0^\infty x^{p-1} e^{-x}\, dx.$$

If p is a natural number then the inequality is proved immediately by the Taylor expansion. The general case can be proved in a minor modification.

Lemma 2.2. *Let $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \infty$. Then the following L^p– $\exp L^2$ estimates hold*:

$$\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} \leqslant \|u_0\|_{\exp L^2(\mathbb{R}^n)} \quad \text{for } u_0 \in \exp L^2(\mathbb{R}^n),\ t > 0, \tag{2.2}$$

$$\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} \leqslant Ct^{-\frac{n}{2p}}\left(\log\left(t^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}}\|u_0\|_{L^p(\mathbb{R}^n)} \quad \text{for } u_0 \in L^p(\mathbb{R}^n),\ t > 0, \tag{2.3}$$

$$\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} \leqslant Ct^{-\frac{n}{2q}}\|u_0\|_{L^q(\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)} \quad \text{for } u_0 \in L^q(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ t > 0. \tag{2.4}$$

Proof. The first inequality is showed by the standard L^p–L^q estimate of the heat kernel and the Taylor expansion. For any $\lambda > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\exp\left(\frac{e^{t\Delta}u_0}{\lambda}\right)^2 - 1\right)dx \\
&\quad = \sum_{k=1}^{\infty}\frac{\|e^{t\Delta}u_0\|_{L^{2k}(\mathbb{R}^n)}^{2k}}{k!\lambda^{2k}} \leqslant \sum_{k=1}^{\infty}\frac{\|u_0\|_{L^{2k}(\mathbb{R}^n)}^{2k}}{k!\lambda^{2k}} = \int_{\mathbb{R}^n}\left(\exp\left(\frac{u_0}{\lambda}\right)^2 - 1\right)dx.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} &= \inf\left\{\lambda > 0;\ \int_{\mathbb{R}^n}\left(\exp\left(\frac{e^{t\Delta}u_0}{\lambda}\right)^2 - 1\right)dx \leqslant 1\right\} \\
&\leqslant \inf\left\{\lambda > 0;\ \int_{\mathbb{R}^n}\left(\exp\left(\frac{u_0}{\lambda}\right)^2 - 1\right)dx \leqslant 1\right\} \\
&= \|u_0\|_{\exp L^2(\mathbb{R}^n)}.
\end{aligned}$$

This proves (2.2). The second inequality can be shown by the similar argument. Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\exp\left(\frac{e^{t\Delta}u_0}{\lambda}\right)^2 - 1\right)dx &= \sum_{k=1}^{\infty}\frac{\|e^{t\Delta}u_0\|_{L^{2k}(\mathbb{R}^n)}^{2k}}{k!\lambda^{2k}} \leqslant \sum_{k=1}^{\infty}\frac{C^{2k}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2k})2k}\|u_0\|_{L^p(\mathbb{R}^n)}^{2k}}{k!\lambda^{2k}} \\
&= t^{\frac{n}{2}}\left(\exp\left(\frac{Ct^{-\frac{n}{2p}}\|u_0\|_{L^p(\mathbb{R}^n)}}{\lambda}\right)^2 - 1\right)
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} &\leqslant \inf\left\{\lambda > 0;\ t^{\frac{n}{2}}\left(\exp\left(\frac{Ct^{-\frac{n}{2p}}\|u_0\|_{L^p(\mathbb{R}^n)}}{\lambda}\right)^2 - 1\right) \leqslant 1\right\} \\
&= Ct^{-\frac{n}{2p}}\left(\log\left(t^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}}\|u_0\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{2.6}$$

This proves (2.3). The inequality (2.4) is proved easily by the standard L^p–L^q estimate and the embedding $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^2(\mathbb{R}^n)$ as

$$\begin{aligned}\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} &\leqslant \left\|e^{t\Delta}u_0\right\|_{L^\infty(\mathbb{R}^n)} + \left\|e^{t\Delta}u_0\right\|_{L^2(\mathbb{R}^n)} \\ &\leqslant Ct^{-\frac{n}{2q}}\|u_0\|_{L^q(\mathbb{R}^n)} + \|u_0\|_{L^2(\mathbb{R}^n)}. \qquad \square \end{aligned} \tag{2.7}$$

Lemma 2.3. *Let $n \geqslant 5, q > \frac{n}{2}$. Then there exists $C = C(n,q) > 0$ such that for every $f \in L^\infty(0,\infty; L^1 \cap L^q(\mathbb{R}^n))$ the following estimate holds*:

$$\left\| \int_0^t e^{(t-s)\Delta} f(s)\,ds \right\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant C\|f\|_{L^\infty(0,\infty;L^1\cap L^q(\mathbb{R}^n))}.$$

Proof. By Lemma 2.2, we obtain

$$\left\|e^{t\Delta}u_0\right\|_{\exp L^2(\mathbb{R}^n)} \leqslant C(t)\big(\|u_0\|_{L^1(\mathbb{R}^n)} + \|u_0\|_{L^q(\mathbb{R}^n)}\big), \tag{2.8}$$

where

$$C(t) = \min\left\{Ct^{-\frac{n}{2q}} + 1, Ct^{-\frac{n}{2}}\big(\log\big(t^{-\frac{n}{2}} + 1\big)\big)^{-\frac{1}{2}}\right\}.$$

We remark that $C(\cdot) \in L^1(0,\infty)$ due to the assumption $n \geqslant 5, q > \frac{n}{2}$. Thus

$$\begin{aligned}\left\| \int_0^t e^{(t-s)\Delta} f(s)\,ds \right\|_{\exp L^2(\mathbb{R}^n)} &\leqslant \int_0^t \left\|e^{(t-s)\Delta} f(s)\right\|_{\exp L^2(\mathbb{R}^n)} ds \\ &\leqslant \int_0^t C(t-s)\big(\left\|f(s)\right\|_{L^1(\mathbb{R}^n)} + \left\|f(s)\right\|_{L^q(\mathbb{R}^n)}\big) \\ &\leqslant \|f\|_{L^\infty(0,\infty;L^1\cap L^q(\mathbb{R}^n))} \int_0^\infty C(s)\,ds \end{aligned} \tag{2.9}$$

for every $t > 0$. This proves Lemma 2.3. $\square$

We remark that Lemma 2.3 does not hold for $1 \leqslant n \leqslant 4$. On the other hand, we show the following property for $n = 4$ instead of Lemma 2.3. Let $\phi(u) := e^{u^2} - 1 - u^2$ and $L^\phi(\mathbb{R}^n)$ be the Orlicz space with the Luxemburg norm

$$\|u\|_{L^\phi(\mathbb{R}^n)} := \inf\left\{\lambda > 0;\ \int_{\mathbb{R}^n} \phi\left(\frac{|u|}{\lambda}\right) dx \leqslant 1\right\}.$$

From the definition, we have

$$C_1\|u\|_{\exp L^2(\mathbb{R}^n)} \leqslant \|u\|_{L^2(\mathbb{R}^n)} + \|u\|_{L^\phi(\mathbb{R}^n)} \leqslant C_2\|u\|_{\exp L^2(\mathbb{R}^n)}$$

for some $C_1, C_2 > 0$. Now we obtain the following estimate similar to Lemma 2.3.

Lemma 2.4. *Let $n = 4$. Then there exists a positive constant $C > 0$ such that for every $f \in L^\infty(0,\infty; L^1 \cap L^4(\mathbb{R}^4))$ the following estimate holds*:

$$\left\| \int_0^t e^{(t-s)\Delta} f(s)\,ds \right\|_{L^\infty(0,\infty;L^\phi(\mathbb{R}^4))} \leqslant C\|f\|_{L^\infty(0,\infty;L^1\cap L^4(\mathbb{R}^4))}. \tag{2.10}$$

Proof. The proof of Lemma 2.4 is similar to the proof of Lemma 2.3. We start to prove

$$\left\| e^{t\Delta}u_0 \right\|_{L^\phi(\mathbb{R}^n)} \leqslant Ct^{-\frac{n}{2}}\left(\log\left(t^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}. \tag{2.11}$$

Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi\left(\frac{|e^{t\Delta}u_0|}{\lambda}\right) dx &\leqslant \sum_{k=2}^\infty \frac{\|e^{t\Delta}u_0\|_{L^{2k}(\mathbb{R}^n)}^{2k}}{\lambda^{2k}k!} \\
&\leqslant \sum_{k=2}^\infty \frac{C^{2k}t^{-\frac{n}{2}(1-\frac{1}{2k})2k}\|u_0\|_{L^1(\mathbb{R}^n)}^{2k}}{\lambda^{2k}k!} \\
&\leqslant t^{\frac{n}{2}}\phi\left(\frac{Ct^{-\frac{n}{2}}\|u_0\|_{L^1(\mathbb{R}^n)}}{\lambda}\right) \\
&\leqslant t^{\frac{n}{2}}\left(\exp\left(\frac{Ct^{-\frac{n}{2}}\|u_0\|_{L^1(\mathbb{R}^n)}}{\lambda}\right)^4 - 1\right).
\end{aligned} \tag{2.12}$$

Therefore we obtain that

$$\begin{aligned}
\left\| e^{t\Delta}u_0 \right\|_{L^\phi(\mathbb{R}^n)} &\leqslant \inf\left\{\lambda > 0;\ t^{\frac{n}{2}}\left(\exp\left(\frac{Ct^{-\frac{n}{2}}\|u_0\|_{L^1(\mathbb{R}^n)}}{\lambda}\right)^4 - 1\right) \leqslant 1\right\} \\
&= Ct^{-\frac{n}{2}}\left(\log\left(t^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{4}}\|u_0\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{2.13}$$

On the other hand, we see that

$$\left\| e^{t\Delta}u_0 \right\|_{L^\phi(\mathbb{R}^4)} \leqslant \left\| e^{t\Delta}u_0 \right\|_{L^\infty(\mathbb{R}^4)} + \left\| e^{t\Delta}u_0 \right\|_{L^4(\mathbb{R}^4)} \leqslant Ct^{-\frac{1}{2}}\|u_0\|_{L^4(\mathbb{R}^4)} + \|u_0\|_{L^4(\mathbb{R}^4)} \tag{2.14}$$

from the embedding $L^4(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow L^\phi(\mathbb{R}^n)$. Combining the inequalities (2.13) and (2.14), we have

$$\left\| e^{t\Delta}u_0 \right\|_{\exp L^\phi(\mathbb{R}^4)} \leqslant C(t)\|u_0\|_{L^1\cap L^4(\mathbb{R}^4)},$$

where

$$C(t) := \min\left\{1 + t^{-\frac{1}{2}}, t^{-2}\left(\log\left(t^{-2}+1\right)\right)^{-\frac{1}{4}}\right\}.$$

We remark that $C(\cdot) \in L^1(0,\infty)$. By the same argument in the proof of Lemma 2.3, we obtain (2.10). $\square$

3. The higher dimensional case ($n \geqslant 5$)

In this section, we prove Theorem 1.3 for $n \geqslant 5$.

Proposition 3.1. *Let $n \geqslant 5$. Then there exists a positive constant $\varepsilon > 0$ such that for all $u_0 \in \exp L^2(\mathbb{R}^n)$ with $\|u_0\|_{\exp L^2(\mathbb{R}^n)} \leqslant \varepsilon$ there exists a solution $u \in L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))$ of (1.6) with (1.4).*

Now we prepare some notations. For $M > 0$,

$$\Phi[u] := e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} f\bigl(u(s)\bigr)\,ds,$$

$$X_M := \bigl\{u \in L^\infty\bigl(0,\infty;\exp L^2\bigl(\mathbb{R}^n\bigr)\bigr);\ \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant M\bigr\}. \tag{3.1}$$

For the proof of Proposition 3.1, we prove that Φ is a contraction map in X_M for a suitable $M > 0$. Therefore we prepare the following estimate of the nonlinear term from Lemma 2.3.

Lemma 3.2. *Let $n \geqslant 5$ and M be a sufficiently small positive number. For every $u, v \in X_M$, the following estimates hold*:

$$\left\| \int_0^t e^{(t-s)\Delta}\bigl(f\bigl(u(s)\bigr)\bigr)\,ds \right\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant H(M)\|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \tag{3.2}$$

and

$$\left\| \int_0^t e^{(t-s)\Delta}\bigl(f\bigl(u(s)\bigr) - f\bigl(v(s)\bigr)\bigr)\,ds \right\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant H(M)\|u-v\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}, \tag{3.3}$$

where $H(M) \to 0$ as $M \to 0$.

Proof. Since (3.2) follows from (3.3) by choosing $v = 0$, we prove only (3.3). From the assumption (1.4), we see

$$\begin{aligned}\bigl|f(u) - f(v)\bigr| &\leqslant C|u-v|\bigl|ue^{\lambda u^2} + ve^{\lambda v^2}\bigr| \\ &= C|u-v|\bigl|u\bigl(e^{\lambda u^2} - 1\bigr) + v\bigl(e^{\lambda v^2} - 1\bigr)\bigr| + C\bigl|u^2 - v^2\bigr|.\end{aligned} \tag{3.4}$$

From Lemma 2.3 and inequality (3.4), we have

$$\begin{aligned}&\left\| \int_0^t e^{(t-s)\Delta}\bigl(f(u) - f(v)\bigr)\,ds \right\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \\ &\quad\leqslant C\bigl\|f(u) - f(v)\bigr\|_{L^\infty(0,\infty;L^1\cap L^p(\mathbb{R}^n))} \\ &\quad\leqslant C\bigl\|(u-v)\bigl(u\bigl(e^{\lambda u^2} - 1\bigr) + v\bigl(e^{\lambda v^2} - 1\bigr)\bigr) + \bigl(u^2 - v^2\bigr)\bigr\|_{L^\infty(0,\infty;L^1\cap L^p(\mathbb{R}^n))}.\end{aligned} \tag{3.5}$$

By Hölder's inequality, we obtain

$$\begin{aligned}
&\left\|(u-v)\left(u\left(e^{\lambda u^2}-1\right)+v\left(e^{\lambda v^2}-1\right)\right)\right\|_{L^p(\mathbb{R}^n)} \\
&\quad \leqslant C\|u-v\|_{L^{2p}(\mathbb{R}^n)}\left(\|u\|_{L^{4p}(\mathbb{R}^n)}\left\|e^{\lambda u^2}-1\right\|_{L^{4p}(\mathbb{R}^n)}+\|v\|_{L^{4p}(\mathbb{R}^n)}\left\|e^{\lambda v^2}-1\right\|_{L^{4p}(\mathbb{R}^n)}\right).
\end{aligned} \tag{3.6}$$

If $M^2 < \frac{1}{4p\lambda}$, then

$$\left\|e^{\lambda u^2}-1\right\|_{L^{4p}(\mathbb{R}^n)} \leqslant \left(\int_{re^n} \exp\left(4p\lambda M^2 \frac{u^2}{\|u\|^2_{\exp L^2(\mathbb{R}^n)}}\right)-1\,dx\right)^{\frac{1}{4p}} \leqslant (4p\lambda)^{\frac{1}{4p}} M^{\frac{1}{2p}}. \tag{3.7}$$

According to the inequalities (3.6), (3.7) and Lemma 2.1, we have

$$\begin{aligned}
\left\|(u-v)\left(u\left(e^{\lambda u^2}-1\right)+v\left(e^{\lambda v^2}-1\right)\right)\right\|_{L^p(\mathbb{R}^n)} &\leqslant C\|u-v\|_{L^{2p}(\mathbb{R}^n)}\left(\|u\|^{1+\frac{1}{2p}}_{\exp L^2(\mathbb{R}^n)}+\|v\|^{1+\frac{1}{2p}}_{\exp L^2(\mathbb{R}^n)}\right) \\
&\leqslant CM^{1+\frac{1}{2p}}\|u-v\|_{\exp L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.8}$$

Let us turn to estimate the term u^2-v^2:

$$\begin{aligned}
\left\|u^2-v^2\right\|_{L^p(\mathbb{R}^n)} &\leqslant \|u-v\|_{L^{2p}(\mathbb{R}^n)}\left(\|u\|_{L^{2p}(\mathbb{R}^n)}+\|v\|_{L^{2p}(\mathbb{R}^n)}\right) \\
&\leqslant C\|u-v\|_{\exp L^2(\mathbb{R}^n)}\left(\|u\|_{\exp L^2(\mathbb{R}^n)}+\|v\|_{\exp L^2(\mathbb{R}^n)}\right) \\
&\leqslant CM\|u-v\|_{\exp L^2(\mathbb{R}^n)}.
\end{aligned} \tag{3.9}$$

For the case $p=1$, we obtain similar estimates to (3.6) and (3.9). We obtain (3.3) immediately from estimates (3.6) and (3.9). $\square$

Proof of Proposition 3.1. We solve the problem (1.6) with the nonlinear term (1.4) by a contraction mapping argument. From Lemmas 2.1 and 3.2, we obtain that Φ is a map on X_M to itself if M and $\|u_0\|_{\exp L^2(\mathbb{R}^n)}$ are small enough. Moreover Φ will be a contraction map on X_M if M is small enough. $\square$

4. The lower dimensional case ($1 \leqslant n \leqslant 4$)

In this section, we prove Theorem 1.3 for $1 \leqslant n \leqslant 4$.

Proposition 4.1. *Let $1 \leqslant n \leqslant 4$. Then there exists a positive constant $\varepsilon > 0$ such that for all $u_0 \in \exp L^2(\mathbb{R}^n)$ with $\|u_0\|_{\exp L^2(\mathbb{R}^n)} \leqslant \varepsilon$ there exists a solution $u \in L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))$ of (1.6) with (1.4).*

We prove Proposition 4.1 by a contraction mapping argument. Let Φ be as above in (3.1) and Y_M be the following:

$$Y_M := \left\{u \in L^\infty\left(0,\infty;\exp L^2\left(\mathbb{R}^n\right)\right);\ \sup_{t>0} t^\sigma \left\|u(t)\right\|_{L^p(\mathbb{R}^n)} + \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant M\right\},$$

where $1+\frac{4}{n} < p < 2+\frac{8}{n}$, $\sigma = \frac{n}{2}(\frac{1}{2}-\frac{1}{p})$. The subspace Y_M is a complete metric space with the distance $d(u,v) := \sup_{t>0} t^\sigma \|u(t)-v(t)\|_{L^p(\mathbb{R}^n)}$. Now we prepare the following estimate for the nonlinear term.

Lemma 4.2. *Let* $1 \leqslant n \leqslant 4$. *For every* $u, v \in Y_M$, *the following estimates hold*:

$$\left\| \int_0^t e^{(t-s)\Delta} f(u)\, ds \right\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))} \leqslant H(M) \tag{4.1}$$

and

$$\sup_{t>0} t^\sigma \left\| \int_0^t e^{(t-s)\Delta} \big(f(u) - f(v)\big)\, ds \right\|_{L^p(\mathbb{R}^n)} \leqslant C(M) \sup_{t>0} t^\sigma \|u - v\|_{L^p(\mathbb{R}^n)}, \tag{4.2}$$

where

$$H(M) = o(M), \qquad C(M) = \begin{cases} O(M^4), & n = 1, \\ O(M^2), & n = 2, 3, \\ O(M), & n = 4. \end{cases}$$

Proof of the inequality (4.2). We prove the inequality (4.2) for $n = 4$. In this case, the nonlinear term satisfies

$$\big|f(u) - f(v)\big| \leqslant C|u - v| \big|ue^{\lambda u^2} + ve^{\lambda v^2}\big| \leqslant C|u - v| \sum_{k=1}^\infty \frac{\lambda^k}{k!} \big(|u|^{2k-1} + |v|^{2k-1}\big)$$

for some $\lambda > 0$, $C > 0$. Let

$$\begin{gathered} \sigma = 1 - \frac{2}{p}, \qquad \rho = 4k, \qquad \theta = \frac{1}{(2k-1)^2}, \\ \frac{1}{q} = \frac{1}{2} - \frac{p-2}{2p(2k-1)}, \qquad \frac{1}{r} = 1 - \frac{k(p-2)}{p(2k-1)}. \end{gathered} \tag{4.3}$$

By the Taylor expansion and the standard L^p–L^q estimate for the heat kernel, we have

$$\begin{aligned} & t^\sigma \left\| \int_0^t e^{(t-s)\Delta} \big(f(u) - f(v)\big)\, ds \right\|_{L^p} \\ & \quad \leqslant C \sum_{k=1}^\infty \frac{\lambda^k}{k!} t^\sigma \int_0^t \big\| e^{(t-s)\Delta} \big(|u - v| \big(|u|^{2k-1} + |v|^{2k-1}\big)\big) \big\|_{L^p}\, ds \\ & \quad \leqslant C \sum_{k=1}^\infty \frac{\lambda^k}{k!} t^\sigma \int_0^t (t-s)^{-2(\frac{1}{r} - \frac{1}{p})} \big\| (u - v)\big(u^{2k-1} + v^{2k-1}\big) \big\|_{L^r}\, ds, \end{aligned} \tag{4.4}$$

where p, r satisfy $2 < p < 4$, $1 \leqslant r \leqslant p$. Applying the Hölder inequality, the Hölder interpolation inequality and Lemma 2.1 we have

$$
\begin{aligned}
&\big\|(u-v)\big(u^{2k-1}+v^{2k-1}\big)\big\|_{L^r}\\
&\quad\leqslant \big\|(u-v)\big\|_{L^p}\big(\|u\|_{L^{q(2k-1)}}^{2k-1}+\|v\|_{L^{q(2k-1)}}^{2k-1}\big)\\
&\quad\leqslant \big\|(u-v)\big\|_{L^p}\big(\|u\|_{L^p}^{(2k-1)\theta}\|u\|_{L^\rho}^{(2k-1)(1-\theta)}+\|v\|_{L^p}^{(2k-1)\theta}\|v\|_{L^\rho}^{(2k-1)(1-\theta)}\big)\\
&\quad\leqslant \Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}}\big\|(u-v)\big\|_{L^p}\\
&\qquad\times\big(\|u\|_{L^p}^{(2k-1)\theta}\|u\|_{\exp L^2}^{(2k-1)(1-\theta)}+\|v\|_{L^p}^{(2k-1)\theta}\|v\|_{\exp L^2}^{(2k-1)(1-\theta)}\big),
\end{aligned}
\tag{4.5}
$$

where the exponents θ, ρ, p, q, r satisfy

$$
0\leqslant\theta\leqslant 1,\qquad 2\leqslant\rho,\qquad \frac{1}{r}=\frac{1}{p}+\frac{1}{q},\qquad \frac{1}{(2k-1)q}=\frac{\theta}{p}+\frac{1-\theta}{\rho}.
$$

Due to the inequalities (4.4) and (4.5), we have

$$
\begin{aligned}
&t^\sigma\left\|\int_0^t e^{(t-s)\Delta}\big(f(u)-f(v)\big)\,ds\right\|_{L^p}\\
&\quad\leqslant C\sum_{k=1}^\infty\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}} t^\sigma\int_0^t (t-s)^{-2(\frac{1}{r}-\frac{1}{p})}s^{-\sigma(1+(2k-1)\theta)}s^\sigma\big\|u(s)-v(s)\big\|_{L^p}\\
&\qquad\times\Big(\big(s^\sigma\|u\|_{L^p}\big)^{(2k-1)\theta}\|u\|_{\exp L^2}^{(2k-1)(1-\theta)}+\big(s^\sigma\|v\|_{L^p}\big)^{(2k-1)\theta}\|v\|_{\exp L^2}^{(2k-1)(1-\theta)}\Big)\,ds.
\end{aligned}
\tag{4.6}
$$

We remark

$$
\begin{aligned}
&\sup_{s>0}s^\sigma\big\|u(s)\big\|_{L^p(\mathbb{R}^n)}\leqslant M,\qquad \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}\leqslant M,\\
&\sup_{s>0}s^\sigma\big\|v(s)\big\|_{L^p(\mathbb{R}^n)}\leqslant M,\qquad \|v\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}\leqslant M,
\end{aligned}
$$

since $u, v\in Y_M$. Therefore we have

$$
\begin{aligned}
&t^\sigma\left\|\int_0^t e^{(t-s)\Delta}\big(f(u)-f(v)\big)\,ds\right\|_{L^p}\\
&\quad= C\sum_{k=1}^\infty\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}} M^{2k-1}t^{1+\sigma-2(\frac{1}{r}-\frac{1}{p})-\sigma(1+(2k-1)\theta)}\\
&\qquad\times\int_0^1(1-s)^{-2(\frac{1}{r}-\frac{1}{p})}s^{-\sigma(1+(2k-1)\theta)}\,ds\times\sup_{s>0}s^\sigma\big\|u(s)-v(s)\big\|_{L^p}\\
&\quad= C\sum_{k=1}^\infty\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}} M^{2k-1}
\end{aligned}
$$

$$\times B\left(1-2\left(\frac{1}{r}-\frac{1}{p}\right), 1-\sigma\left(1+(2k-1)\theta\right)\right) \times \sup_{s>0} s^{\sigma} \left\|u(s)-v(s)\right\|_{L^p}, \tag{4.7}$$

where B is the beta function. We remark that the exponents p, q, r, θ, ρ in (4.3) satisfy

$$\begin{gathered} 2<p<4, \qquad 1 \leqslant r \leqslant p, \qquad -1<-2\left(\frac{1}{r}-\frac{1}{p}\right), \qquad -1<-\sigma\left(1+(2k-1)\theta\right), \\ 0 \leqslant \theta \leqslant 1, \qquad 2 \leqslant \rho, \qquad \frac{1}{r}=\frac{1}{p}+\frac{1}{q}, \qquad 1+\sigma-2\left(\frac{1}{r}-\frac{1}{p}\right)-\sigma\left(1+(2k-1)\theta\right)=0, \\ \frac{1}{(2k-1)q}=\frac{\theta}{p}+\frac{1-\theta}{\rho}. \end{gathered} \tag{4.8}$$

Since $B(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$, we have that

$$B\left(1-2\left(\frac{1}{r}-\frac{1}{p}\right), 1-\sigma\left(1+(2k-1)\theta\right)\right)=\frac{\Gamma(\frac{2}{p})}{\Gamma(\frac{p-2}{p(2k-1)})\Gamma(\frac{2}{r}-1)}. \tag{4.9}$$

From $\Gamma(a)=O(1/a)$ as $a \to 0$, we have $\Gamma(\frac{p-2}{p(2k-1)})=O(k)$ as $k \to \infty$. Thus,

$$B\left(1-2\left(\frac{1}{r}-\frac{1}{p}\right), 1-\sigma\left(1+(2k-1)\theta\right)\right) \leqslant C \tag{4.10}$$

for any $k \in \mathbb{N}$. Applying Stirling's formula, we have

$$\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}} \leqslant C\left(\frac{\rho}{2}+1\right)^{k} \leqslant C^k k! \tag{4.11}$$

for some $C>0$. Combining (4.7), (4.10) and (4.11) we have

$$\begin{aligned} & t^{\sigma}\left\|\int_0^t e^{(t-s)\Delta}\left(f(u)-f(v)\right)ds\right\|_{L^p} \\ & \quad \leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k-1)(1-\theta)}{\rho}} M^{2k-1} \\ & \qquad \times B\left(1-2\left(\frac{1}{r}-\frac{1}{p}\right), 1-\sigma\left(1+(2k-1)\theta\right)\right)\sup_{s>0} s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p} \\ & \quad \leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}M^{2k-1}C^k k! \sup_{s>0} s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p} \\ & \quad \leqslant CM\sup_{s>0} s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p} \end{aligned} \tag{4.12}$$

for small M. This proves (4.2) for $n=4$.

We prove (4.2) for $n=2,3$ by a minor change of the proof for the case $n=4$ above. Let

$$\rho=\frac{2(p-2)nk-2(2p-a)}{a-4},\qquad \theta=\frac{2p-a}{nk(p-2)},\qquad \frac{1}{q}=\frac{a}{np},\qquad \frac{1}{r}=\frac{1}{p}+\frac{a}{np} \tag{4.13}$$

for some $4<a<np-n$. We remark that $4<np-n$ since $1+\frac{4}{n}<p$. From the Taylor expansion, Hölder's inequality and the standard L^p–L^q estimate for the heat kernel, we have

$$\begin{aligned}
&t^\sigma\left\|\int_0^t e^{(t-s)\Delta}\bigl(f(u)-f(v)\bigr)\,ds\right\|_{L^p}\\
&\quad\leqslant C\sum_{k=1}^\infty\frac{\lambda^k}{k!}t^\sigma\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\bigl\|(u-v)\bigl(u^{2k}+v^{2k}\bigr)\bigr\|_{L^r}\,ds\\
&\quad\leqslant C\sum_{k=1}^\infty\frac{\lambda^k}{k!}t^\sigma\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\bigl\|(u-v)\bigr\|_{L^p}\bigl(\|u\|_{L^{q(2k)}}^{2k}+\|v\|_{L^{q(2k)}}^{2k}\bigr)\,ds.
\end{aligned}\tag{4.14}$$

Applying Hölder's interpolation inequality,

$$\|u\|_{L^{q(2k)}}\leqslant\|u\|_{L^p}^{\theta}\|u\|_{L^\rho}^{1-\theta},\qquad \frac{1}{q(2k)}=\frac{\theta}{p}+\frac{1-\theta}{\rho} \tag{4.15}$$

and the inequality (2.1), we have

$$\begin{aligned}
&t^\sigma\left\|\int_0^t e^{(t-s)\Delta}\bigl(f(u)-f(v)\bigr)\,ds\right\|_{L^p}\\
&\quad\leqslant C\sum_{k=1}^\infty\frac{\lambda^k}{k!}t^\sigma\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\bigl\|(u-v)\bigr\|_{L^p}\\
&\qquad\times\bigl(\|u\|_{L^p}^{2k\theta}\|u\|_{L^\rho}^{2k(1-\theta)}+\|v\|_{L^p}^{2k\theta}\|v\|_{L^\rho}^{2k(1-\theta)}\bigr)\,ds\\
&\quad\leqslant C\sum_{k=1}^\infty\frac{\lambda^k}{k!}t^\sigma\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}\bigl\|(u-v)\bigr\|_{L^p}\\
&\qquad\times\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}\bigl(\|u\|_{L^p}^{2k\theta}\|u\|_{\exp L^2}^{2k(1-\theta)}+\|v\|_{L^p}^{2k\theta}\|v\|_{\exp L^2}^{2k(1-\theta)}\bigr)\,ds.
\end{aligned}\tag{4.16}$$

Now we recall

$$\begin{aligned}
&\sup_{s>0}s^\sigma\bigl\|u(s)\bigr\|_{L^p(\mathbb{R}^n)}\leqslant M,\qquad \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}\leqslant M,\\
&\sup_{s>0}s^\sigma\bigl\|v(s)\bigr\|_{L^p(\mathbb{R}^n)}\leqslant M,\qquad \|v\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}\leqslant M.
\end{aligned}\tag{4.17}$$

Applying (4.17) to (4.16), we see that

$$
\begin{aligned}
& t^{\sigma}\left\|\int_0^t e^{(t-s)\Delta}\big(f(u)-f(v)\big)\,ds\right\|_{L^p} \\
& \leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}M^{2k} \\
& \quad\times t^{\sigma}\left(\int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{p})}s^{-\sigma(1+2k\theta)}\,ds\right)\sup_{s>0}s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p} \\
& = C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}M^{2k} \\
& \quad\times B\left(1-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right),1-\sigma(1+2k\theta)\right)\sup_{s>0}s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p},
\end{aligned} \tag{4.18}
$$

where the exponents p, q, r, θ, ρ in (4.13) satisfy

$$
\begin{gathered}
1+\frac{4}{n}<p<2+\frac{8}{n},\qquad 1\leqslant r\leqslant p,\qquad -1<-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right),\qquad -1<-\sigma(1+2k\theta), \\
0\leqslant\theta\leqslant 1,\qquad 2\leqslant\rho,\qquad \frac{1}{r}=\frac{1}{p}+\frac{1}{q},\qquad 1+\sigma-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right)-\sigma(1+2k\theta)=0, \\
\frac{1}{2kq}=\frac{\theta}{p}+\frac{1-\theta}{\rho}.
\end{gathered} \tag{4.19}
$$

For those exponents, we obtain that

$$
B\left(1-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{p}\right),1-\sigma(1+2k\theta)\right)=\frac{\Gamma(1-\sigma)}{\Gamma(1-\frac{n}{2q})\Gamma(\frac{(n+4)p-2n-2a}{4p})}\leqslant C \tag{4.20}
$$

and

$$
\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}\leqslant C\left(\frac{\rho}{2}+1\right)^k\leqslant C^k k! \tag{4.21}
$$

for some positive constant $C>0$. Combining (4.18), (4.20) and (4.21) we have

$$
\begin{aligned}
& t^{\sigma}\left\|\int_0^t e^{(t-s)\Delta}\big(f(u)-f(v)\big)\,ds\right\|_{L^p} \\
& \leqslant \sum_{k=1}^{\infty}C\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}M^{2k} \\
& \quad\times B\left(1-2\left(\frac{1}{r}-\frac{1}{p}\right),1-\sigma(1+2k\theta)\right)\times\sup_{s>0}s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p}
\end{aligned}
$$

$$\leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}M^{2k}C^k k!\sup_{s>0}s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p}$$
$$\leqslant CM^2\sup_{s>0}s^{\sigma}\left\|u(s)-v(s)\right\|_{L^p} \tag{4.22}$$

for small M. This proves (4.2) for $n=2,3$. $\square$

Proof of the inequality (4.1). We prove (4.1) for $n=4$. Let $\phi(u):=e^{u^2}-1-u^2$ and $L^{\phi}(\mathbb{R}^n)$ be the Orlicz space with the Luxemburg norm

$$\|u\|_{L^{\phi}(\mathbb{R}^n)}:=\inf\left\{\lambda>0;\ \int_{\mathbb{R}^n}\phi\left(\frac{|u|}{\lambda}\right)dx\leqslant 1\right\}.$$

From the definition, we have

$$C_1\|u\|_{\exp L^2(\mathbb{R}^n)}\leqslant\|u\|_{L^2(\mathbb{R}^n)}+\|u\|_{L^{\phi}(\mathbb{R}^n)}\leqslant C_2\|u\|_{\exp L^2(\mathbb{R}^n)}$$

for some $C_1,C_2>0$. Therefore it is enough to prove (4.1) for $n=4$ by proving

$$\left\|\int_0^t e^{(t-s)\Delta}f(u)\,ds\right\|_{L^{\infty}(0,\infty;L^2(\mathbb{R}^n))}\leqslant H(M) \tag{4.23}$$

and

$$\left\|\int_0^t e^{(t-s)\Delta}f(u)\,ds\right\|_{L^{\infty}(0,\infty;L^{\phi}(\mathbb{R}^n))}\leqslant H(M). \tag{4.24}$$

We start to prove (4.23). Let

$$\rho=4k,\qquad \theta=\frac{1}{(2k-1)},\qquad \frac{1}{r}=1-\frac{k(1-\frac{2}{p})}{2k-1}. \tag{4.25}$$

By the Taylor expansion, Hölder's inequality, Hölder's interpolation inequality and the standard L^p–L^q estimate for the heat kernel, we have

$$\left\|\int_0^t e^{(t-s)\Delta}f(u)\,ds\right\|_{L^2(\mathbb{R}^n)}$$
$$\leqslant\sum_{k=1}^{\infty}\frac{1}{k!}\int_0^t C(t-s)^{-2(\frac{1}{r}-\frac{1}{2})}\|u\|_{L^p(\mathbb{R}^n)}^{2k\theta}\|u\|_{L^{\rho}(\mathbb{R}^n)}^{2k(1-\theta)}\,ds$$
$$\leqslant\sum_{k=1}^{\infty}\frac{1}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}}\int_0^t C(t-s)^{-2(\frac{1}{r}-\frac{1}{2})}\|u\|_{L^p(\mathbb{R}^n)}^{2k\theta}\|u\|_{\exp L^2(\mathbb{R}^n)}^{2k(1-\theta)}\,ds$$

$$\leqslant \sum_{k=1}^{\infty} \frac{1}{k!} C\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}} M^{2k} B\left(-2\left(\frac{1}{r}-\frac{1}{2}\right)+1, -2k\theta\sigma+1\right), \tag{4.26}$$

where p, r, θ, ρ in (4.25) satisfy

$$2 < p < 4, \qquad 1 \leqslant r \leqslant p, \qquad -1 < -2\left(\frac{1}{r}-\frac{1}{2}\right), \qquad -1 < -2k\theta\sigma, \qquad 0 \leqslant \theta \leqslant 1,$$
$$2 \leqslant \rho, \qquad -2\left(\frac{1}{r}-\frac{1}{2}\right) - 2k\theta\sigma + 1 = 0, \qquad \frac{1}{2kr} = \frac{\theta}{p} + \frac{1-\theta}{\rho}. \tag{4.27}$$

For those exponents, we obtain that

$$B\left(-2\left(\frac{1}{r}-\frac{1}{2}\right)+1, -2k\theta\sigma+1\right) = \frac{\Gamma(1)}{\Gamma(\frac{2k(p-2)}{p(2k-1)})\Gamma(1-\frac{2k(p-2)}{p(2k-1)})} \leqslant C \tag{4.28}$$

and

$$\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}} \leqslant C^k k!. \tag{4.29}$$

Combining (4.26), (4.28) and (4.29) we have

$$\begin{aligned}\left\| \int_0^t e^{(t-s)\Delta} f(u)\,ds \right\|_{L^2(\mathbb{R}^n)} &\leqslant \sum_{k=1}^{\infty} \frac{C}{k!} \Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}} M^{2k} B\left(-2\left(\frac{1}{r}-\frac{1}{2}\right)+1, -2k\theta\sigma+1\right) \\ &\leqslant \sum_{k=1}^{\infty} \frac{C}{k!} C^k k! M^{2k} \leqslant CM^2. \end{aligned} \tag{4.30}$$

This proves (4.23).

We turn to prove (4.24). By the same argument in the proof of Lemma 3.2, we obtain (4.24). Here we use Lemma 2.4 instead of Lemma 2.3.

Let us turn to prove (4.1) for $n = 2, 3$. We apply a different argument as in the proof of (4.1) for $n = 4$. Let $1 \leqslant q \leqslant 2$ and a be the smallest positive number satisfying $a = 2\log(a+1)$. We see that

$$\left(\log\left((t-s)^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}} \leqslant \sqrt{2}(t-s)^{\frac{n}{4}} \quad \text{for } 0 \leqslant s \leqslant t - a^{-\frac{2}{n}}.$$

Therefore we have

$$\begin{aligned}&\left\| \int_0^t e^{(t-s)\Delta} f\big(u(s)\big)\,ds \right\|_{\exp L^2(\mathbb{R}^n)} \\ &\quad \leqslant \int_0^t (t-s)^{-\frac{n}{2q}} \left(\log\left((t-s)^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}} \left\| f(u) \right\|_{L^q} ds\end{aligned}$$

$$\leqslant \int_0^{t-a^{-\frac{2}{n}}} (t-s)^{-\frac{n}{2q}} \left(\log\left((t-s)^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}} \left\| f(u) \right\|_{L^q} ds$$

$$+ \int_{t-a^{-\frac{2}{n}}}^{t} (t-s)^{-\frac{n}{2q}} \left(\log\left((t-s)^{-\frac{n}{2}}+1\right)\right)^{-\frac{1}{2}} \left\| f(u) \right\|_{L^q} ds$$

$$\leqslant \sqrt{2} \int_0^t (t-s)^{-\frac{n}{2q}+\frac{n}{4}} \left\| f(u) \right\|_{L^q} ds + \sup_{t>0} C \left\| f(u) \right\|_{L^q} =: I + II. \tag{4.31}$$

Hereafter we estimate I and II separately. The first part I is proved by the Taylor expansion, Hölder's inequality, Hölder's interpolation inequality and the standard L^p–L^q estimate for the heat kernel. In this case, we recall that the nonlinear term satisfies

$$\left| f(u) \right| \leqslant C|u|^3 e^{\lambda u^2} = C \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} |u|^{2k+1}$$

for some $\lambda > 0$, $C > 0$. Let

$$\rho = \frac{2n((2k+1)(p-2)-ap)}{(n+4)(p-2)-nap}, \qquad \theta = \frac{ap}{(p-2)(2k+1)}, \qquad \frac{1}{q} = \frac{1}{2} + \frac{2}{n} - \frac{a}{2}, \tag{4.32}$$

for some $1 < a < \frac{4}{n}$. One can choose such an a since $1 \leqslant n \leqslant 3$. For those exponents we have

$$I \leqslant C \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-\frac{n}{2q}+\frac{n}{4}} \|u\|_{L^{(2k+1)q}}^{2k+1} ds$$

$$\leqslant C \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \int_0^t (t-s)^{-\frac{n}{2q}+\frac{n}{4}} \|u\|_{L^p}^{(2k+1)\theta} \Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k+1)(1-\theta)}{\rho}} \|u\|_{\exp L^2}^{(2k+1)(1-\theta)} ds$$

$$\leqslant C \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k+1)(1-\theta)}{\rho}} M^{2k+1} \int_0^t (t-s)^{-\frac{n}{2q}+\frac{n}{4}} s^{-(2k+1)\theta\sigma} ds$$

$$= C \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k+1)(1-\theta)}{\rho}} M^{2k+1} t^{1-\frac{n}{2q}+\frac{n}{4}-(2k+1)\theta\sigma} B\left(1-\frac{n}{2q}+\frac{n}{4}, 1-(2k+1)\theta\sigma\right),$$

where q, θ, ρ in (4.32) satisfy

$$\frac{n}{2} < q < 2, \qquad -1 < -\frac{n}{2q} + \frac{n}{4}, \qquad -1 < -(2k+1)\sigma\theta, \qquad 0 \leqslant \theta \leqslant 1,$$

$$2 \leqslant \rho, \qquad 1 - \frac{n}{2q} + \frac{n}{4} - (2k+1)\sigma\theta = 0, \qquad \frac{1}{(2k+1)q} = \frac{\theta}{p} + \frac{1-\theta}{\rho}. \tag{4.33}$$

For those exponents, we obtain that

$$
\begin{aligned}
B\left(1-\frac{n}{2q}+\frac{n}{4},1-(2k+1)\theta\sigma\right) &= \frac{\Gamma(1)}{\Gamma(1-\frac{n}{2q}+\frac{n}{4})\Gamma(1-(2k+1)\theta\sigma)} \\
&\leqslant \frac{\Gamma(1)}{\Gamma(1-\frac{n}{2q}+\frac{n}{4})\Gamma(1-\frac{ap}{4})} \\
&\leqslant C
\end{aligned}
\tag{4.34}
$$

and

$$
\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{2k(1-\theta)}{\rho}} \leqslant C^k k!. \tag{4.35}
$$

Combining (4.33), (4.34) and (4.35) we have

$$
\begin{aligned}
I &\leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\Gamma\left(\frac{\rho}{2}+1\right)^{\frac{(2k+1)(1-\theta)}{\rho}} M^{2k+1}B\left(1-\frac{n}{2q}+\frac{n}{4},1-(2k+1)\theta\sigma\right) \\
&\leqslant C\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}M^{2k+1}C^k k! \leqslant CM^3
\end{aligned}
\tag{4.36}
$$

for small M.

The second part II in (4.31) can be proved in the same argument in the proof of (3.2). This proves (4.1) for $n=2,3$.

We now comment on the case $n=1$. The method in the proof above can be applied with minor modifications. Namely, we assume that

$$
f(u)=\left(e^{u^2}-1-u^2\right)u=\sum_{k=2}^{\infty}\frac{u^{2k+1}}{k!}
$$

so that the exponent ρ in (4.32) is greater than 2, i.e. $\exp L^2(\mathbb{R}^n)\hookrightarrow L^\rho(\mathbb{R}^n)$. Indeed,

$$
\rho=\begin{cases} \frac{2((2k+1)(p-2)-ap)}{5(p-2)-ap}\geqslant 2 & \text{for } k\geqslant 2, \\ \frac{2(3(p-2)-ap)}{5(p-2)-ap}\leqslant 2 & \text{for } k=1. \end{cases} \qquad \Box
$$

Proof of Proposition 4.1. We solve the problem (1.6) with the nonlinear term (1.4) by a contraction mapping argument. By Lemmas 2.1 and 4.2, we obtain that Φ is a map on Y_M to itself if M and $\|u_0\|_{\exp L^2(\mathbb{R}^n)}$ are small enough. Moreover, Φ will be a contraction map on Y_M if M is small enough. $\Box$

5. Continuity to the initial data

In this section, we prove the continuities to the initial data (1.5), (1.7) and (1.8). For the proof of (1.5), we introduce a rearrangement function. Let u^* be the non-increasing rearrangement of a function u defined by

$$
u^*(r):=\inf\left\{\lambda>0;\ \mu_u(\lambda)\leqslant r\right\},
$$

where $\mu_u(\lambda)$ is a distribution function of u, i.e. $\mu_u(\lambda):=|\{x\in\mathbb{R}^n;\ |u(x)|>\lambda\}|$. Let u^{**} be the average function of u^*, namely

$$u^{**}(r) := \frac{1}{r} \int_0^r u^*(\eta)\, d\eta.$$

We state some basic properties of the rearrangement.

Proposition 5.1. *Let u be a measurable function, u^* be the rearrangement of u and u^{**} be the average function of u^*. Then u^* and u^{**} have the following properties*:

(1) (*Non-increasing*) *If $r_1 \leqslant r_2$, then $u^*(r_1) \geqslant u^*(r_2)$.*
(2) (*Non-negative*) *$u^*(r) \geqslant 0$ for any $0 < r$.*
(3) *For any $0 < r$, $u^*(r) \leqslant u^{**}(r)$.*
(4) *If $\Phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function which satisfies $\Phi(0) = 0$, then*

$$\int_{\mathbb{R}^n} \Phi\bigl(\bigl|u(x)\bigr|\bigr)\, dx = \int_0^\infty \Phi\bigl(\bigl|u^*(r)\bigr|\bigr)\, dr.$$

(5) *The triangle inequality for the average function holds,*

$$(u+v)^{**}(r) \leqslant u^{**}(r) + v^{**}(r).$$

For the proof of Proposition 5.1, see [11]. Next lemma is important for the proof of (1.5).

Lemma 5.2. *(See [7,9].) Let u^* be the rearrangement of u and u^{**} be the average function of u^*. Then the following equivalence holds*:

$$C_1 \|u\|_{\exp L^2(\mathbb{R}^n)} \leqslant \sup_{0<r<1} \frac{u^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} + \|u\|_{L^2(\mathbb{R}^n)} \leqslant C_2 \|u\|_{\exp L^2(\mathbb{R}^n)}$$

for some $C_1, C_2 > 0$.

Let us turn to prove (1.5).

Proof. Here we prove only

$$\sup_{0<r<1} \frac{u^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} \leqslant C \|u\|_{\exp L^2(\mathbb{R}^n)}. \tag{5.1}$$

For the proof of Lemma 5.2, see [7,9]. Let $K^{**} := \sup_{0<r<1} \frac{u^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}}$ and we prove $K^{**} \leqslant C\|u\|_{\exp L^p}$. Applying Jensen's inequality to the convex function $x \mapsto e^{x^2} - 1$ and Proposition 5.1, we have

$$\begin{aligned}
\exp\biggl(\frac{u^{**}(r)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\biggr)^2 - 1 &= \exp\biggl(\frac{1}{r}\int_0^r \frac{u^*(\eta)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\, d\eta\biggr)^2 - 1 \\
&\leqslant \frac{1}{r}\int_0^r \biggl(\exp\biggl(\frac{u^*(\eta)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\biggr)^2 - 1\biggr)\, d\eta
\end{aligned}$$

$$\leqslant \frac{1}{r}\int_0^\infty \left(\exp\left(\frac{u^*(\eta)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\right)^2 - 1\right) d\eta$$

$$= \frac{1}{r}\int_{\mathbb{R}^n} \left(\exp\left(\frac{u(x)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\right)^2 - 1\right) dx.$$

From the definition of the Luxemburg norm, we have

$$\int_{\mathbb{R}^n} \left(\exp\left(\frac{u(x)}{\|u\|_{\exp L^2(\mathbb{R}^n)}}\right)^2 - 1\right) dx \leqslant 1.$$

Thus we see that

$$u^{**}(r) \leqslant \|u\|_{\exp L^2(\mathbb{R}^n)} \left(\log\left(\frac{1}{r} + 1\right)\right)^{1/2}.$$

We obtain that

$$\sup_{0<r<1} \frac{u^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} \leqslant \sup_{0<r<1} \frac{\|u\|_{\exp L^2(\mathbb{R}^n)} (\log(\frac{1}{r}+1))^{\frac{1}{2}}}{(\log \frac{e}{r})^{\frac{1}{2}}} \leqslant C\|u\|_{\exp L^2(\mathbb{R}^n)}. \qquad \square$$

Proof of the inequality (1.5). According to Lemma 5.2 and the triangle inequality of the mean value $u^{**}(r) := \frac{1}{r}\int_0^r u^*(\eta)\, d\eta$ (Proposition 5.1), we have

$$\begin{aligned}
\left\|e^{t\Delta}u_0 - u_0\right\|_{\exp L^2(\mathbb{R}^n)} &\geqslant C \sup_{0<r<1} \frac{(e^{t\Delta}u_0 - u_0)^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} \\
&\geqslant C \lim_{r\to 0} \frac{(e^{t\Delta}u_0 - u_0)^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} \\
&\geqslant C \lim_{r\to 0} \frac{(u_0)^{**}(r) - (e^{t\Delta}u_0)^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}}.
\end{aligned}$$

We remark that $(e^{t\Delta}u_0)^{**} \in L^\infty(0,\infty)$ for all $t > 0$ since $e^{t\Delta}u_0 \in L^\infty(\mathbb{R}^n)$ for all $t > 0$. Therefore we see that

$$\lim_{r\to 0} \frac{(e^{t\Delta}u_0)^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}} = 0. \tag{5.2}$$

Thus we have

$$\left\|e^{t\Delta}u_0 - u_0\right\|_{\exp L^2(\mathbb{R}^n)} \geqslant C \lim_{r\to 0} \frac{(u_0)^{**}(r)}{(\log \frac{e}{r})^{\frac{1}{2}}}.$$

Let C_n be the measure of the unit ball in $\mathbb{R}^n$. If we choose

$$u_0(x) := \begin{cases} (\log \frac{e}{C_n|x|^n})^{\frac{1}{2}} - \frac{1}{2}(\log \frac{e}{C_n|x|^n})^{-\frac{1}{2}} & \text{for } 0 < |x| < 1, \\ 0 & \text{otherwise}, \end{cases} \tag{5.3}$$

one may obtain easily that $(u_0)^{**}(r) = (\log \frac{e}{r})^{\frac{1}{2}}$ for $0 < r < C_n$. Therefore we obtain

$$\left\| e^{t\Delta} u_0 - u_0 \right\|_{\exp L^2(\mathbb{R}^2)} \geqslant 1$$

for the initial data (5.3). This completes the proof. $\square$

Proof of the properties (1.8) and (1.7). We prove (1.8). Let $p > \frac{n}{2}$. From the embedding relation $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^2(\mathbb{R}^n)$ and the standard L^p–L^q estimate for the heat kernel, we have

$$\begin{aligned} \left\| u(t) - e^{t\Delta} u_0 \right\|_{\exp L^2(\mathbb{R}^n)} &\leqslant \int_0^t \left\| e^{(t-s)\Delta} f(u) \right\|_{\exp L^2(\mathbb{R}^n)} ds \\ &\leqslant \int_0^t \left\| e^{(t-s)\Delta} f(u) \right\|_{L^2(\mathbb{R}^n)} + \left\| e^{(t-s)\Delta} f(u) \right\|_{L^\infty(\mathbb{R}^n)} ds \\ &\leqslant \int_0^t \left\| f(u) \right\|_{L^2(\mathbb{R}^n)} + (t-s)^{-\frac{n}{2p}} \left\| f(u) \right\|_{L^p(\mathbb{R}^n)} ds. \end{aligned} \tag{5.4}$$

Let us calculate $\|f(u)\|_{L^p(\mathbb{R}^n)}$. By the same argument in (3.7), we have if $M^2 \leqslant \frac{1}{\lambda p}$,

$$\left\| f(u) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \left(\int_{\mathbb{R}^n} e^{\lambda p u(t,x)^2} - 1 \, dx \right)^{\frac{1}{p}} \leqslant C(\lambda p)^{\frac{1}{p}} \left\| u(t) \right\|_{\exp L^2(\mathbb{R}^n)}^{\frac{2}{p}}. \tag{5.5}$$

From (5.4) and (5.5) we obtain

$$\begin{aligned} \left\| u(t) - e^{t\Delta} u_0 \right\|_{\exp L^2(\mathbb{R}^n)} &\leqslant \int_0^t \left\| u(t) \right\|_{\exp L^2(\mathbb{R}^n)}^{\frac{1}{2}} + (t-s)^{-\frac{n}{2p}} \left\| u(t) \right\|_{\exp L^2(\mathbb{R}^n)}^{\frac{2}{p}} ds \\ &\leqslant \int_0^t \left\| u(t) \right\|_{\exp L^2(\mathbb{R}^n)}^{\frac{1}{2}} + (t-s)^{-\frac{n}{2p}} \left\| u(t) \right\|_{\exp L^2(\mathbb{R}^n)}^{\frac{2}{p}} ds \\ &\leqslant t \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}^{\frac{1}{2}} + t^{1-\frac{n}{2p}} \|u\|_{L^\infty(0,\infty;\exp L^2(\mathbb{R}^n))}^{\frac{2}{p}} \\ &\to 0 \quad \text{as } t \to 0. \end{aligned}$$

This completes the proof of (1.8).

We next prove (1.7). Let X be the pre-dual space of $\exp L^2(\mathbb{R}^n)$ (i.e. $X^* = \exp L^2(\mathbb{R}^n)$). It is known that X is a Banach space and $C_0^\infty(\mathbb{R}^n)$ is dense in X (cf. [1]). Let φ be in X. By Hölder's inequality for the Orlicz space (cf. [1]), we have

$$\left|\int_{\mathbb{R}^n}\left(e^{t\Delta}u_0(x)-u_0(x)\right)\varphi(x)\,dx\right|=\left|\int_{\mathbb{R}^n}u_0(x)\left(e^{t\Delta}\varphi(x)-\varphi(x)\right)dx\right|$$
$$\leqslant 2\|u_0\|_{\exp L^2(\mathbb{R}^n)}\left\|e^{t\Delta}\varphi-\varphi\right\|_X.$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in X, we have $\lim_{t\to 0}\|e^{t\Delta}\varphi-\varphi\|_X=0$. This completes the proof of (1.7). □

Remark 5.3. One can prove Theorem 1.3 for the nonlinear term $f(u)=|u|^{\frac{4}{n}}ue^{u^2}$ by the same argument in the proof of the case $n=4$. We mention that $|f(u)|=O(|u|^{1+\frac{4}{n}})$ as $|u|\to 0$. This order is critical in the sense of the global solvability for the heat equation (1.2) in $L^2(\mathbb{R}^n)$.

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